

Fault-tolerant quantum error correction code conversion

Charles D. Hill, Austin G. Fowler, David S. Wang, and Lloyd C. L. Hollenberg
*Centre for Quantum Computation and Communication Technology,
School of Physics, University of Melbourne, Melbourne, Victoria 3010, Australia.*

In this paper we demonstrate how data encoded in a five-qubit quantum error correction code can be converted, fault-tolerantly, into a seven-qubit Steane code. This is achieved by progressing through a series of codes, each of which fault-tolerantly corrects at least one error. Throughout the conversion the encoded qubit remains protected. We found, through computational search, that the method used to convert between codes given in this paper is optimal.

PACS numbers: 03.67.Pp, 03.67.Ac

I. INTRODUCTION

In order for a quantum computer to be able to achieve large scale quantum computing, quantum error correction codes (QECC) will be required to mitigate the effects of errors due to decoherence and/or control imprecision. Without quantum error correction (QEC), the detrimental effects of bit flips and phase flips on data qubits can quickly render a quantum computer useless. Many different quantum error correction codes are known [1, 3, 5–8]. Each error correction code has different advantages and disadvantages. The five-qubit code [7] is the smallest QEC code able to correct an arbitrary single-qubit error. Although it is a good candidate for quantum memory protection, it is difficult to manipulate data encoded in the five-qubit error correction code in a fault-tolerant manner. This is because there are no transversal multiple-qubit logical gates, and few transversal single-qubit gates. In contrast, the seven-qubit Steane code [6, 8] is able to perform a variety of transversal logical gates, including a transversal CNOT gate, and several transversal single-qubit rotations. However, the seven-qubit code is less efficient than the five-qubit code at storing information, using two additional data qubits to encode a single logical qubit.

There are situations in which it is desirable to change the encoding of qubits stored by a quantum computer. For example, it may be desirable to have information stored in long term memory encoded using one error correction code, and the information being manipulated by the processor encoded using a different error correction code. The requirements for memory might be focused on using *small* codes, whereas those for the processor focused on ease of operation. Similarly, information being transmitted down a bus might be optimized to mitigate errors due to loss, but this might not be a relevant requirement for a processor. It is therefore necessary to have efficient, fault-tolerant methods to convert between different error correction codes.

In this paper, we demonstrate how data encoded in one quantum error correction code can be fault-tolerantly converted into another. Specifically, we show how data encoded in a five-qubit QEC code can be converted, efficiently and fault-tolerantly, into a seven-qubit Steane

code.

Conversion between error correction codes is achieved by progressing through a series of codes, each of which is a valid error correction code in its own right. Each of these codes is only slightly different from the last. Each code can not only correct any single qubit error, but also any extra errors which might be introduced by the conversion operations. If error correction is applied at each step of the conversion, the encoded information remains protected.

There are many different ways to convert between codes. The method we present in this paper was found through computational search to use the fewest number of two-qubit control-sign (CZ) operations to convert between codes. We obtained the optimal conversion between the two codes by performing a breadth-first search from codes locally equivalent to the five-qubit code and proceeded to codes locally equivalent to the seven-qubit code. This allowed us to find a path of minimal length between the two codes.

This paper is organized as follows: Section II describes the method used to convert between codes. The QEC codes in this section are enumerated in the appendix. Section III considers the initial and final codes, and demonstrates that they are the five- and seven-qubit codes respectively. Discussion of the computational search is given in Section IV and the resources required are discussed in Section V. Finally, a conclusion is given in Section VI.

II. METHOD

This section describes the conversion from the five-qubit code to the seven-qubit code. We assume that the information is initially encoded in the five-qubit code, and we would like this information to be encoded in the seven-qubit code. For the purpose of this paper, we allow a total of ten data qubits for storing the data — three ancilla qubits above the seven needed to encode data in the seven-qubit code, and five more than is required for the five-qubit code. Initially, each of these five ancilla qubits is assumed to be initialized in the $|+\rangle$ state. In addition to these ten data qubits, additional qubits are

required to fault-tolerantly determine the syndrome measurements. These qubits will be discussed in more detail in Section V.

Throughout this paper, we use X , Y and Z to refer to the Pauli matrices σ_X , σ_Y and σ_Z , as well as I to refer to the identity. In writing out the stabilizers for a multi-qubit code or state, we omit the tensor product (e.g. $X \otimes X$ is written as XX). The Hadamard gate is defined as $H = (X + Z)/\sqrt{2}$. We will also generically refer to the generators of a given stabilizer group for a given code as ‘the stabilizers’ of the code.

In order to convert between codes, we assume that we are able to perform two basic operations. These are:

Application of the CZ Gate: We allow the application of the controlled-sign gate between any two qubits. This well known gate is a two-qubit entangling operation and applies a π phase to the $|11\rangle$ state, while not affecting any of the other states. That is

$$CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (1)$$

Single-qubit operations: Single-qubit operations (in particular, the Hadamard gate, and X and Z rotations) can be applied in parallel to the qubits.

In order to convert a five-qubit code to a seven-qubit code, several different operations are performed, giving rise to slightly different QEC codes at each step. At each step, we will explicitly consider both the stabilizers of the code, and the logical operations. A circuit of the operations which are applied to convert from the five-qubit code to the seven-qubit code is shown in Figure 1. After each operation, appropriate stabilizers are fault-tolerantly measured and, if required, corrections may be applied.

If a CZ gate is applied, it introduces the possibility of errors on two different qubits. One situation where this can happen is when a single qubit error occurs during the operation of the CZ gate. For our purposes, we assume the worst possible scenario: that two-qubit gate can cause any combination of one or two Pauli errors on the qubits affected by the gate. In order to be fault-tolerant, therefore, we ensure the subsequent code is able to identify and correct both one-qubit errors (on any qubit), and two-qubit errors on both ends of an applied CZ gate.

The application of a stabilizer does not affect the state at all. Therefore if two errors differ by only a stabilizer of a state, their effect on the state is equivalent. Similarly, they may be corrected by applying the same operation to the quantum state. We do not need to be able to distinguish such errors, and refer to the two errors as equivalent ‘modulo the stabilizers’. Errors which are not equivalent modulo the stabilizers, we will call ‘distinct’.

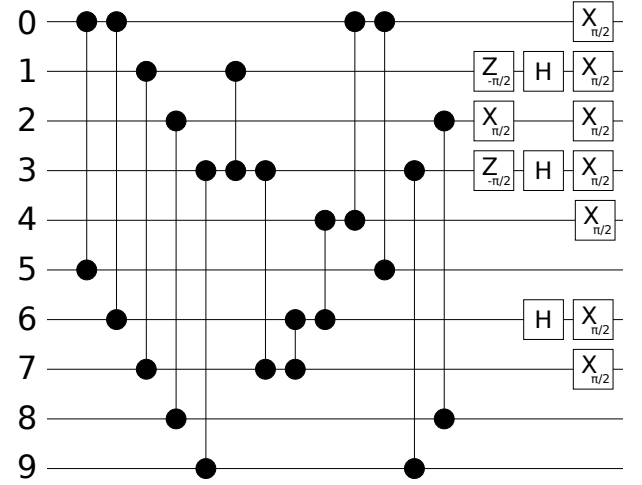


FIG. 1. Circuit diagram of conversion between error correction codes. Solid circles with a line connecting them represent control-sign (CZ) gates. Single-qubit rotations are labelled by the axis of rotation and a subscripted angle of rotation. Hadamard gates are labelled by ‘ H ’.

It is possible to verify that each of these codes is a valid error correction code. This is achieved by enumerating all errors and verifying that they produce a unique syndrome, or are equivalent errors modulo the stabilizers. The sets of stabilizers for each code (enumerated in the appendix) were checked (by computer) and verified to give a unique syndrome for all distinct sets of errors. Both single qubit errors and two-qubit errors were considered.

The final QEC code is locally equivalent to the seven-qubit Steane code. Information which was initially encoded in the five-qubit error correction code has been fault-tolerantly transferred to be protected by the seven-qubit Steane code.

III. LOCAL EQUIVALENCE OF THE CODES

In this section, we show that the initial and final codes are locally equivalent to the five- and seven-qubit codes respectively. The initial and final codes are not expressed in the usual sets of stabilizers found in the literature for five- and seven-qubit codes. Here we explicitly give the local operations which convert between the forms used in the paper.

The initial code is locally equivalent to the five-qubit code. Its stabilizers are similar to the traditional five-qubit code [3]:

Stabilizers				
Y	Y	Z	I	Z
Z	Y	Y	Z	I
I	Z	Y	Y	Z
Z	I	Z	Y	Y

The logical operators for the five-qubit QEC code are:

Logical Operators					
$X_L =$	X	X	X	X	X
$Z_L =$	Z	Z	Z	Z	Z

The only difference is a change from X to Y and qubit reordering, which has no effect other than to change which syndrome is associated with which error. For the purposes of this paper, we use this trivially changed version of the error correction code.

In the final stage of the code conversion, a set of single-qubit rotations is applied. There is no reason why these operations cannot be applied in parallel. This operation is fault-tolerant.

The final code is slightly more difficult to verify. After the completion of the CZ gates and the first three local operations (not including the Hadamard gate), the code is as follows:

Stabilizers									
X	I	X	Z	I	I	Z	I	I	I
I	Z	X	I	X	I	Z	I	I	I
Z	X	Z	I	I	I	I	Z	I	I
I	I	Z	X	Z	I	I	Z	I	I
I	I	I	I	I	X	I	I	I	I
Z	I	I	I	Z	I	X	Z	I	I
I	Z	I	Z	I	I	Z	X	I	I
I	I	I	I	I	I	I	I	X	I
I	I	I	I	I	I	I	I	I	X

The logical operators for the QEC code are:

Logical Operators									
$X_L =$	Y	X	X	X	Y	I	I	I	I
$Z_L =$	Z	Z	Y	Z	Z	I	I	I	I

After the application of the Hadamard gates to qubits 1, 3 and 6, and the removal of unentangled qubits 9, 8 and 5, the stabilizers of the code become:

Stabilizers						
X	I	X	X	I	X	I
I	X	X	I	X	X	I
I	X	I	X	I	X	X
Z	Z	Z	I	I	I	Z
I	I	Z	Z	Z	I	Z
Z	I	I	I	Z	Z	Z

with the following logical operators,

Logical Operators							
$X_L =$	X	X	X	X	X	X	X
$Z_L =$	Y	Y	Y	Y	Y	Y	Y

The code stabilizers are made up of only combinations of Z and X terms alone. Rearranging the order of these terms, and the qubits

$$\begin{aligned} 0 &\leftarrow 0 \\ 1 &\leftarrow 1 \\ 2 &\leftarrow 6 \\ 3 &\leftarrow 2 \\ 4 &\leftarrow 4 \\ 5 &\leftarrow 3 \\ 6 &\leftarrow 5 \end{aligned}$$

and rearranging the stabilizers gives:

Stabilizers						
X	X	X	X	I	I	I
X	X	I	I	X	X	I
X	I	X	I	X	I	X
Z	Z	Z	Z	I	I	I
Z	Z	I	I	Z	Z	I
Z	I	Z	I	Z	I	Z

with the following logical operators,

Logical Operators							
$X_L =$	X	X	X	X	X	X	X
$Z_L =$	Y	Y	Y	Y	Y	Y	Y

Although the logical Z_L operation (at this point) is not the standard $Z^{\otimes 7}$, this can easily be remedied by applying an $X_{\pi/2}$ gate to each of the seven qubits. After this operation, the stabilizers remain the same, but the logical operators become:

Logical Operators							
$X_L =$	X	X	X	X	X	X	X
$Z_L =$	Z	Z	Z	Z	Z	Z	Z

IV. OPTIMAL SOLUTION

Figure 1 is an optimal solution — that is, it uses the fewest number of two-qubit operations. In order to find this solution, we used a computer program, which searched through the space of possibilities as follows:

Starting at both the five and the seven-qubit codes, the program exhaustively enumerates all locally equivalent graph states corresponding to the encoded logical operators of each code [2, 4]. The search then proceeded by applying all possible CZ gates to each of these codes. Each resulting code is then checked to see if it is a valid quantum error correction code, and only accepted if it is able to correct any relevant error at that point. The algorithm proceeds as a breadth-first search until the first collision is found between codes originating from both

the five- and seven-qubit codes. When such a collision is found, we have found a shortest length path from the five to the seven-qubit codes.

In order to keep the number of error correction codes searched to a manageable size, it was important to identify codes which are isomorphic to each other. Many codes are isomorphic simply because of a rearrangement of qubits. Although, in general, identifying all isomorphic codes is difficult, some codes can be quickly be recognised as isomorphic to each other. These were cached, and not searched twice.

Similarly, each path through a series of codes is isomorphic to other QEC codes by multiplication of single-qubit rotations. These operations do not change the distance of the error correction code, and so only one such representative path was considered.

Using this method we were able to find paths of shortest length between the five-qubit and seven-qubit QEC codes.

The solution we found was restricted to use exactly ten data qubits. For differing numbers of available data qubits, different solutions might be possible. For example, a solution with eleven, or nine qubits might be obtainable.

Often we will also want to convert the seven-qubit code into the five-qubit code. In general, it is not possible to simply reverse the order of a set of steps converting one code to another. This is because of the extra two-qubit errors which need to be accounted for when two-qubit operations are applied. In the reverse direction it is a *different* set of stabilizers, and a different error correction code which has to account for these extra errors. For this particular set of codes, it is possible to run all the steps in reverse direction as well: the error correction code (in the reverse direction) can also account for the relevant two-qubit errors. Since our solution was found only checking one of the two directions (away from the five-qubit code for the first half of the operations or away from the seven-qubit codes for the second), the solution we found is the optimum solution for the more stringent requirement that the conversion be reversible.

V. RESOURCES REQUIRED

We now consider the resources required to fault-tolerantly transfer a five-qubit code to a seven-qubit code. In the procedure above, we explicitly used codes of up to ten data qubits to encode the information. In addition to these, ancilla qubits are required to make syndrome measurements. In the scheme we have presented, stabilizers of up to weight 6 need to be measured (although there are stabilizers of weight eight listed in the appendix, combinations of these are equivalent to stabilizers of weight six). This then requires seven additional qubits, six of which are prepared in a cat state, in order to perform fault-tolerant measurement of the syndrome [7]. In total then, at any one time we require a maximum of 17 qubits including ten data qubits and seven ancilla qubits for measurement of the syndrome.

Including only operations modifying the error correction codes (as opposed to counting adding or removing qubits, syndrome measurement and error correction), the number of operations required to convert between codes is 15, made up of 14 CZ gates, and one application of single-qubit gates in parallel (3 Hadamard gates, 8 X rotations and 2 Z rotations).

VI. CONCLUSION

We have shown a new, optimal, method to fault-tolerantly convert between the five-qubit and seven-qubit Steane QEC codes. The conversion works by changing through a series of quantum error correction codes, each slightly distinct from the last. Each of these codes can correct both single-qubit errors, as well as any two-qubit errors which might be introduced by two-qubit operations. We have shown that each step was valid by explicitly calculating the stabilizers of the code, and verifying that every syndrome produced (modulo the stabilizers of the code) is unique.

This research was conducted by the Australian Research Council Centre of Excellence for Quantum Computation and Communication Technology (project number CE110001027), with support from the US National Security Agency and the US Army Research Office under contract number W911NF-08-1-0527.

-
- [1] S. B. Bravyi and A. Yu. Kitaev. Quantum codes on a lattice with boundary. *Quantum Computers and Computing*, 2(1):43–48, 2001. quant-ph/9811052.
 - [2] Andrew Cross, Graeme Smith, John A. Smolin, and Bei Zeng. Codeword stabilized quantum codes. *IEEE Trans. Info. Theory*, 55, 2009.
 - [3] D. Gottesman. *Stabilizer codes and quantum error correction*. PhD thesis, California Institute of Technology, 1997.
 - [4] M. Hein, J. Eisert, and H. J. Briegel. Multiparty entanglement in graph states. *Phys. Rev. A*, 69(6):062311–, June 2004.
 - [5] E. Knill. Quantum computing with realistically noisy devices. *Nature*, 434(7029):39–44, March 2005.
 - [6] R. Laflamme, C. Miquel, J. P. Paz, and W. H. Zurek. Perfect quantum error correcting code. *Phys. Rev. Lett.*, 77(1):198–201, July 1996.
 - [7] P. W. Shor. Scheme for reducing decoherence in quantum computer memory. *Phys. Rev. A*, 52(4):R2493–R2496, October 1995.

- [8] A. M. Steane. Error correcting codes in quantum theory. *Phys. Rev. Lett.*, 77(5):793–797, July 1996.

APPENDIX: EXPLICIT CODES

In this appendix we explicitly list the stabilizers and the logical operators for each of the codes.

The following operations were performed:

- ### 1. Initial state

The following are the stabilizers for the QEC code:

Stabilizers				
Y	Y	Z	I	Z
Z	Y	Y	Z	I
I	Z	Y	Y	Z
Z	I	Z	Y	Y

The logical operators for the QEC code are:

Logical Operators					
$X_L =$	X	X	X	X	X
$Z_L =$	Z	Z	Z	Z	Z

2. Apply a CZ operation between qubits 0 and 5.

The following are the stabilizers for the QEC code:

[illegible]

The logical operators for the QEC code are:

Logical Operators									
$X_L =$	X	X	X	X	X	Z	I	I	I
$Z_L =$	Z	Z	Z	Z	Z	I	I	I	I

3. Apply a CZ operation between qubits 0 and 6.

The following are the stabilizers for the QEC code:

[illegible]

The logical operators for the QEC code are:

Logical Operators										
$X_L =$	X	X	X	X	X	Z	Z	I	I	I
$Z_L =$	Z	Z	Z	Z	Z	I	I	I	I	I

4. Apply a CZ operation between qubits 1 and 7.

The following are the stabilizers for the QEC code:

Stabilizers									
Y	Y	Z	I	Z	Z	Z	Z	I	I
Z	Y	Y	Z	I	I	I	Z	I	I
I	Z	Y	Y	Z	I	I	I	I	I
Z	I	Z	Y	Y	I	I	I	I	I
Z	I	I	I	I	X	I	I	I	I
Z	I	I	I	I	I	X	I	I	I
I	Z	I	I	I	I	I	X	I	I
I	I	I	I	I	I	I	I	X	I
I	I	I	I	I	I	I	I	I	X

The logical operators for the QEC code are:

Logical Operators										
$X_L =$	X	X	X	X	X	Z	Z	Z	I	I
$Z_L =$	Z	Z	Z	Z	Z	I	I	I	I	I

5. Apply a CZ operation between qubits 2 and 8.

The following are the stabilizers for the QEC code:

[illegible]

The logical operators for the QEC code are:

Logical Operators									
$X_L =$	X	X	X	X	X	Z	Z	Z	I
$Z_L =$	Z	Z	Z	Z	Z	I	I	I	I

6. Apply a CZ operation between qubits 3 and 9.

The following are the stabilizers for the QEC code:

Stabilizers									
Y	Y	Z	I	Z	Z	Z	Z	I	I
Z	Y	Y	Z	I	I	I	Z	Z	I
I	Z	Y	Y	Z	I	I	I	Z	Z
Z	I	Z	Y	Y	I	I	I	I	Z
Z	I	I	I	I	X	I	I	I	I
Z	I	I	I	I	I	X	I	I	I
I	Z	I	I	I	I	I	X	I	I
I	I	Z	I	I	I	I	I	X	I
I	I	I	Z	I	I	I	I	I	X

The logical operators for the QEC code are:

Logical Operators									
$X_L =$	X	X	X	X	X	Z	Z	Z	Z
$Z_L =$	Z	Z	Z	Z	Z	I	I	I	I

7. Apply a CZ operation between qubits 1 and 3.

The following are the stabilizers for the QEC code:

Stabilizers									
Y	Y	Z	Z	Z	Z	Z	Z	I	I
Z	Y	Y	I	I	I	I	Z	Z	I
I	I	Y	Y	Z	I	I	I	Z	Z
Z	Z	Z	Y	Y	I	I	I	I	Z
Z	I	I	I	I	X	I	I	I	I
Z	I	I	I	I	I	X	I	I	I
I	Z	I	I	I	I	I	X	I	I
I	I	Z	I	I	I	I	I	X	I
I	I	I	Z	I	I	I	I	I	X

The logical operators for the QEC code are:

Logical Operators									
$X_L =$	X	Y	X	Y	X	Z	Z	Z	Z
$Z_L =$	Z	Z	Z	Z	Z	I	I	I	I

8. Apply a CZ operation between qubits 7 and 3.

The following are the stabilizers for the QEC code:

Stabilizers									
Y	Y	Z	Z	Z	Z	Z	Z	I	I
Z	Y	Y	I	I	I	I	Z	Z	I
I	I	Y	Y	Z	I	I	Z	Z	Z
Z	Z	Z	Y	Y	I	I	Z	I	Z
Z	I	I	I	I	X	I	I	I	I
Z	I	I	I	I	I	X	I	I	I
I	Z	I	Z	I	I	I	X	I	I
I	I	Z	I	I	I	I	I	X	I
I	I	I	Z	I	I	I	I	I	X

The logical operators for the QEC code are:

Logical Operators										
$X_L =$	X	Y	X	Y	X	Z	Z	I	Z	Z
$Z_L =$	Z	Z	Z	Z	Z	I	I	I	I	I

9. Apply a CZ operation between qubits 7 and 6.

The following are the stabilizers for the QEC code:

Stabilizers									
Y	Y	Z	Z	Z	Z	Z	Z	I	I
Z	Y	Y	I	I	I	I	Z	Z	I
I	I	Y	Y	Z	I	I	Z	Z	Z
Z	Z	Z	Y	Y	I	I	Z	I	Z
Z	I	I	I	I	X	I	I	I	I
Z	I	I	I	I	I	X	Z	I	I
I	Z	I	Z	I	I	Z	X	I	I
I	I	Z	I	I	I	I	I	X	I
I	I	I	Z	I	I	I	I	I	X

The logical operators for the QEC code are:

Logical Operators										
$X_L =$	X	Y	X	Y	X	Z	Z	I	Z	Z
$Z_L =$	Z	Z	Z	Z	Z	I	I	I	I	I

10. Apply a CZ operation between qubits 6 and 4.

The following are the stabilizers for the QEC code:

Stabilizers									
Y	Y	Z	Z	Z	Z	Z	Z	I	I
Z	Y	Y	I	I	I	I	Z	Z	I
I	I	Y	Y	Z	I	I	Z	Z	Z
Z	Z	Z	Y	Y	I	Z	Z	I	Z
Z	I	I	I	I	X	I	I	I	I
Z	I	I	I	Z	I	X	Z	I	I
I	Z	I	Z	I	I	Z	X	I	I
I	I	Z	I	I	I	I	I	X	I
I	I	I	Z	I	I	I	I	I	X

The logical operators for the QEC code are:

Logical Operators										
$X_L =$	Y	X	X	X	Y	I	I	I	I	I
$Z_L =$	Z	Z	Y	Z	Z	I	I	I	I	I